Estimation of Local Vertical and Orbital Parameters for an Earth Satellite Using Horizon Sensor Measurements

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This paper discusses the application of the Kalman filter to the problem of determining the direction of the normal to the earth's surface from aboard an earth satellite, making use of horizon sensor measurements. While providing an optimal filter for noisy horizon sensor measurements, the linear estimation scheme corrects for errors in the assumed values of those elliptical orbit parameters that determine the motion in the orbital plane. A computer program has been written to simulate the actual elliptical motion as well as perform the estimation. Results have been obtained regarding the effects of noise magnitude, frequency of measurements, and choice of actual and nominal parameters on the convergence of the scheme and the improvement it provides in the accuracy of the determination of local vertical.

I. Introduction

THIS paper discusses the application of the Kalman filter in its discrete form to the problem of determining the direction of local vertical from aboard an earth satellite, making use of horizon sensor measurements. In the process of filtering the noisy horizon sensor measurements, the computational scheme corrects for errors in the assumed values of those orbital parameters that determine the motion in the orbital plane.

The use of onboard measurements in conjunction with an optimal filtering procedure for determining vehicle position, velocity, and orientation is not a new concept and has been discussed in the literature for application to space vehicle guidance[1] as well as satellite orbit determination.[2-4] In most cases, deviations in the position and velocity coordinates of the vehicle have been chosen as the state variables of the estimation scheme and matrix inversion has been required as part of the procedure. The present scheme is distinguished by the following features:

1) The state variables are constant in time, or at most slowly varying over several orbits. They consist of three orbital parameters determining the motion of the vehicle center of gravity and two parameters related to the orientation of the vehicle within the orbit plane. A static estimation scheme is therefore involved, requiring no numerical integration for the calculation of transition matrices, etc. The choice of the initial reference orbit is also not very critical, within certain limits.

2) The use of the recursion formulas of the Kalman estimator eliminates the need for matrix inversion. The form of the equations remains the same whether they are used as a least-squares filter (assuming initially that all values of errors in the state variables are equally likely and measurements are uncorrelated) or whether a priori knowledge is included in the form of an initial covariance matrix of errors in assumed parameter values.

3) The computational procedure has been programed for a digital computer and has been tested in conjunction with a simulation of the actual orbital dynamics. We have demonstrated the convergence of the scheme and its ability to predict the direction of local vertical to within an angular accuracy at least an order of magnitude better than the standard deviation of the measurement noise of the horizon sensor. The program allows for studies of the effects of frequency of measurements, frequency of updating, nonlinearities in the orbital equations, choice of initial covariance matrix, etc.

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2. Assumptions and Computational Equations

Much of the simplicity of the scheme results from the assumption that all measurements are taken in the orbital plane and that the vehicle rotation is about an axis normal to the plane. After injection into orbit, an initial stabilization scheme [part of an attitude control system (ACS)] essentially removes rotations about all vehicle axes except one. The ACS is also used to align this axis with the normal to the orbital plane.[5] Although the control problem is not being considered here, it should be mentioned that periodic corrections will have to be applied to realign the axis of rotation because of perturbations in the total angular momentum vector due to small disturbances and the secular perturbations in Ω, the longitude of the ascending node. These effects are relatively small over several orbits. In the following discussion, then, it is possible to assume a vehicle whose axis of rotation is normal to the orbital plane. The orbital elements in the plane are also taken as constants in the initial phase of the discussion, although perturbations in the mean anomaly will be discussed below.

If a vehicle-fixed reference direction is chosen in the plane of rotation, each horizon sensor measurement may be interpreted as an angle between local vertical and this reference direction. The simplest situation to consider is when the reference is in the direction of local vertical at perigee and the satellite is rotating at the mean motion of the orbit ($\omega_s = 2\pi/\tau$, where τ is the period). At any time t after perigee passage, the angle α between local vertical and reference is simply the difference between the true anomaly φ and the mean anomaly M of the elliptical orbit (see Fig. 1):

$$\alpha(t) = \varphi(t) - M(t) \equiv \beta(t) \qquad (1)$$

More generally, for any constant rate of rotation and an arbitrary injection point ($t = 0$), $\alpha(t)$ is given by

$$\alpha(t) = \alpha_b + \beta(t) - \beta(0) - [\omega_s - (2\pi/\tau)]t \qquad (2)$$

At $t = 0$, the angle is just a bias angle α_b, depending upon the arbitrary choice of the reference direction. Thereafter, $\alpha(t)$ depends upon the behavior of $\beta(t)$ as compared to $\beta(0)$, and the difference between the actual rotational rate‡ and the mean motion of the orbit. For orbits of small eccentricity e, $\beta(t)$ may be expressed as a series in e and $M(t)$.[6] $M(t)$ is explicitly determined in terms of T_0, the (virtual) time from perigee passage to the injection point, the period τ, and the time from injection t. Kepler's third law relates τ to the semimajor axis of the ellipse a:

$$\beta(t) = (2e - \tfrac{1}{4}e^3 + \tfrac{5}{96}e^5 + \tfrac{107}{4608}e^7)\sin M(t) + \ldots + \tfrac{47273}{32256}e^7 \sin 7M(t) \quad (3)$$

$$M(t) = 2\pi(T_0 + t)/\tau \quad (4)$$

$$\tau = Ca^{3/2} \quad (5)$$

Symbolically, then, $\alpha(t)$ may be expressed as a function of five independent parameters and time t, as shown in Eq. (6):

$$\alpha(t) = f(e, a, T_0, \omega_s, \alpha_b, t) \quad (6)$$

where the first three are orbital parameters determining the motion of the vehicle center of gravity, and the last two relate to the orientation of the vehicle. The orbital parameters determining the orientation of the orbital plane in inertial space and the orientation of the ellipse in the orbital plane cannot be determined by horizon sensor measurements alone.

The estimation scheme requires the choice of a nominal set of the five parameters from which a predicted angle is calculated:

$$\alpha_{\text{nom}}(t) = f(e_{\text{nom}}, a_{\text{nom}}, T_{0_{\text{nom}}}, \omega_{s_{\text{nom}}}, \alpha_{b_{\text{nom}}}, t) \quad (7)$$

At discrete times along the orbit t_i the predicted angle $\alpha_{\text{nom}}(t_i)$ is compared to the measured angle α_m determined from a noisy horizon sensor measurement of local vertical:

$$\alpha_m(t_i) = \alpha(t_i) + v(t_i) \quad (8)$$

where

$$\mathcal{E}[v(t_i)] = 0$$
$$\mathcal{E}[v(t_i)^2] = \sigma^2 \quad (9)$$
$$\mathcal{E}[v(t_i)v(t_j)] = 0 \qquad i \neq j$$

$$\Delta\alpha_m(t_i) = \alpha_m(t_i) - \alpha_{\text{nom}}(t_i)$$
$$\equiv \Delta\alpha(t_i) + v(t_i) \quad (10)$$

‡ The assumption of a constant rotational rate for a satellite is not strictly valid because of the presence of disturbance torques acting on the vehicle. For a vehicle that is nonsymmetrical about its axis of rotation, the gravity gradient effect produces the major disturbance torque. Disturbance torques arise from radiation pressure and electric and magnetic fields. The magnitudes of these and other torques are discussed in Ref. 7. If the resulting angular acceleration can be described analytically, it may be included in Eq. (2) by replacing $\omega_s t$ with $\int_0^t \omega_s dt$. In the gravity gradient case, for example, ω_s will be a function of $\omega_s(0)$, α_b, and τ, to first order, with higher-order terms in the eccentricity. Since it is extremely difficult to accurately represent all disturbances analytically, a more attractive method consists of measuring the inertial rotational rate with a device such as a celestial drift meter[8] that makes use of the star background. The accuracy of such an instrument should permit its use without the smoothing required for horizon sensor measurements and would eliminate ω_s from among the parameters to be estimated. These schemes require a limited amount of numerical integration in the computations. Occasional application of correction torques will maintain ω_s to within a small variation from a desired constant value. The general development below is applicable to these schemes. In the simulation, a rotationally symmetrical vehicle was assumed, and ω_s was among the estimated parameters.

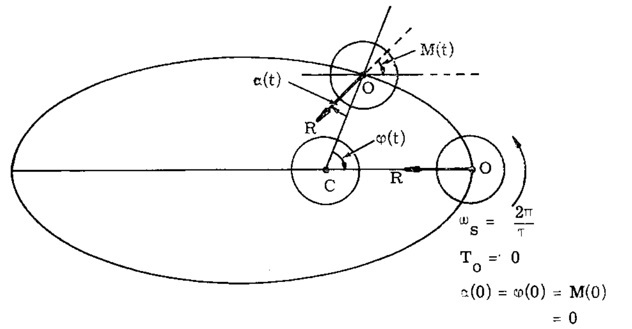

Fig. 1 Significant angles for two satellite positions in the orbital plane: OR = reference direction; OC = local vertical.

The random noise is assumed to be white Gaussian with zero mean. (With the present limited measurement, a bias in the horizon sensor will be interpreted as a spurious error in the parameter α_b.) Linear estimation techniques may then be used if $\Delta\alpha$ is approximated as a linear function of the error parameters $\Delta e = e - e_{\text{nom}}, \ldots, \Delta\alpha_b = \alpha_b - \alpha_{b_{\text{nom}}}$. These error parameters become the state variables of the estimation scheme. The following linearized deviational equation will be valid for small values of the error parameters; the partials are determined analytically from Eqs. (1–5):

$$\Delta\alpha = \left(\frac{\partial\alpha}{\partial e}\right)_{e_{\text{nom}}} \cdot \Delta e + \left(\frac{\partial\alpha}{\partial a}\right)_{a_{\text{nom}}} \cdot \Delta a + \ldots$$

$$\Delta\alpha = \mathbf{h}^T \mathbf{\Delta x}$$

where

$$\mathbf{h} = \begin{bmatrix} \dfrac{\partial\alpha}{\partial e} \\ \dfrac{\partial\alpha}{\partial a} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad \mathbf{\Delta x} = \begin{bmatrix} \Delta e \\ \Delta a \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad (11)$$

3. Linearity Considerations

A limited study has been made of the magnitude of the error parameters in e, a, and T_0 for which the deviational equation is a good approximation. A nominal orbit with $e = 0.10$, $a = 1.16$ ($\tau \approx 105$ min), and $T_0 = 0$ was chosen. $\Delta\alpha$ was calculated from the deviational equation and the exact equations for errors in each parameter individually at three points along the orbit between $\varphi = 0$ and $\varphi = \pi/2$. Table 1 shows some results at the point on the orbit for which the nonlinearity was greatest. Results for a nominal orbit with low eccentricity ($e = 0.01$) are also shown. The error quantities are normalized in terms of the nominal values of the parameters, except for T_0, which is normalized in terms of the orbital period. The results show that the nonlinearity is least significant for errors in e, and greatest for errors in T_0. For a given Δe, the percent nonlinearity is fairly independent of the nominal e in the region considered. This is true for the other parameters as well.

It is possible to calculate the error in the orbital parameters based upon an error in the tangential injection velocity to the nominal orbit. A 1% velocity magnitude error leads to values of $\Delta e/e = 17\%$, $\Delta a/a = 2.3\%$, and $\Delta T_0/\tau = 1.8\%$ for the nominal $e = 0.10$. The percent nonlinearity due to ΔT_0 will therefore be substantial at the point shown in Table 1.

The value of ΔT_0 increases tenfold for a 1% velocity magnitude error at injection into the low eccentricity orbit with $e = 0.01$. The estimation scheme accounts for these non-

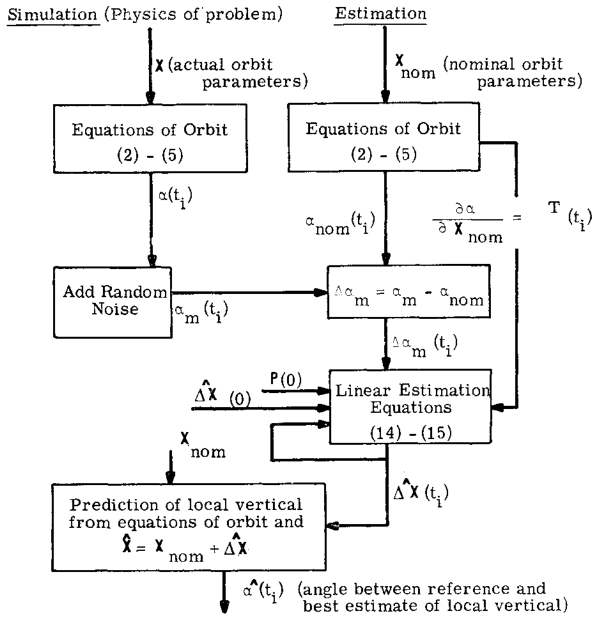

Fig. 2 Flow diagram of computer program.

linear effects by frequent updating of the nominal orbital parameters until the errors are well within the linear region.

4. Linear Estimation Scheme

The estimation problem may be formulated as follows[9,10]: A set of K measurements defines the vector-matrix equation,

$$\mathbf{\Delta z} = \mathbf{A\Delta x} + \mathbf{b} \tag{12}$$

where

$$\mathbf{\Delta z} = \begin{bmatrix} \Delta\alpha_m(t_1) \\ \Delta\alpha_m(t_2) \\ \cdot \\ \cdot \\ \cdot \\ \Delta\alpha_m(t_K) \end{bmatrix}, \quad \mathbf{\Delta x} = \begin{bmatrix} \Delta e \\ \Delta a \\ \cdot \\ \cdot \\ \cdot \\ \Delta\alpha_b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} v(t_1) \\ \cdot \\ \cdot \\ \cdot \\ v(t_K) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \frac{\partial\alpha(t_1)}{\partial e} & \frac{\partial\alpha(t_1)}{\partial a} & \cdots & \frac{\partial\alpha(t_1)}{\partial\alpha_b} \\ \cdots & \cdots & & \cdots \\ \frac{\partial\alpha(t_K)}{\partial e} & \frac{\partial\alpha(t_K)}{\partial a} & \cdots & \frac{\partial\alpha(t_K)}{\partial\alpha_b} \end{bmatrix} = \begin{bmatrix} \mathbf{h}^T(t_1) \\ \cdot \\ \cdot \\ \mathbf{h}^T(t_K) \end{bmatrix} \equiv \begin{bmatrix} \mathbf{h}^T(1) \\ \cdot \\ \cdot \\ \mathbf{h}^T(K) \end{bmatrix} \tag{13}$$

Several investigators[9] have shown that an optimal estimate for $\mathbf{\Delta x}$ after the Kth measurement is given by the following recursion formulas:

$$\mathbf{\Delta\hat{x}}(K) = \mathbf{\Delta\hat{x}}(K-1) + \mathbf{P}(K)\mathbf{h}(K)[\Delta\alpha_m(K) - \mathbf{h}^T(K)\mathbf{\Delta\hat{x}}(K-1)] \tag{14}$$

$$\mathbf{P}(K) = \mathbf{P}(K-1) - \mathbf{P}(K-1)\mathbf{h}(K) \times \{\mathbf{h}^T(K)\mathbf{P}(K-1)\mathbf{h}(K) + 1\}^{-1}\cdot\mathbf{h}^T(K)\mathbf{P}(K-1) \tag{15}$$

$\mathbf{\Delta\hat{x}}(K)$ represents the maximum-likelihood estimate of $\mathbf{\Delta x}$ for the case of Gaussian noise. The recursion formula (15) is equivalent to the more usual form,

$$\mathbf{P}^{-1}(K) = \mathbf{P}^{-1}(K-1) + \mathbf{h}(K)\mathbf{h}^T(K) \tag{16}$$

$$[\mathbf{P}^{-1}(K) = \mathbf{P}^{-1}(0) + \mathbf{h}(1)\mathbf{h}^T(1) + \mathbf{h}(2)\mathbf{h}^T(2) + \ldots + \mathbf{h}(K)\mathbf{h}^T(K) = \mathbf{P}^{-1}(0) + \mathbf{A}^T\mathbf{A}]$$

for the case of a positive definite matrix $\mathbf{P}(K)$. In Eq. (15), however, the need for matrix inversion is avoided because the quantity in braces is a scalar. The equations are a discrete version of the Kalman estimation formulas for the case of constant state variables with noise in the measurements only. If $\mathbf{\Delta x}$ is considered as a Gaussian random vector about which a priori information is known at the initiation of the estimation process on the basis of previous measurements or knowledge of injection errors, etc., then Eq. (17) defines the a priori density function, and Eq. (18) defines the Gaussian a posteriori density function after the measurements $\mathbf{\Delta z}$ have been taken:

$$\mathcal{E}(\mathbf{\Delta x}) = \mathbf{\Delta\hat{x}}(0), \quad \mathrm{cov}[\mathbf{\Delta x} - \mathbf{\Delta\hat{x}}(0)] = \mathbf{P}(0)\cdot\sigma^2 \tag{17}$$

$$\mathcal{E}(\mathbf{\Delta x}/\mathbf{\Delta z}) = \mathbf{\Delta\hat{x}}(K), \quad \mathrm{cov}\{[\mathbf{\Delta x} - \mathbf{\Delta\hat{x}}(K)]/\mathbf{\Delta z}\} = \mathbf{P}(K)\cdot\sigma^2 \tag{18}$$

By extracting the variance of the measurement noise σ^2 from the initial $\mathbf{P}$ matrix [Eq. (17)], a standard recursion formula for the $\mathbf{P}$ matrix is obtained which is independent of the noise level. If

$$\mathbf{\Delta\hat{x}}(0) \to 0$$

$$\mathbf{P}(0) = \begin{bmatrix} R & & 0 \\ & \ddots & \\ 0 & & R \end{bmatrix} \qquad R \to \infty \tag{19}$$

i.e., as the weight given a priori information approaches zero, $\mathbf{\Delta\hat{x}}(K)$ in Eq. (14) approaches the least-squares fit to a set of N parameters based upon K measurements (where $K > N$).

$$\mathbf{\Delta\hat{x}}(K) \to \mathbf{P}(K)\mathbf{A}^T\mathbf{\Delta z} \to (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{\Delta z} \tag{20}$$

The primary purpose of the estimation scheme is the determination of local vertical at all points along the orbit to a

Table 1 Effects of nonlinearity

	$e_{nom} = 0.1$		$e_{nom} = 0.01$	
Normalized error parameter	Magnitude,[a] %	Percent[b] nonlinearity	Magnitude, %	Percent nonlinearity
$\Delta e/e_{nom}$	10	0.89	100	0.90
	20	1.82	200	1.77
$\Delta a/a_{nom}$	1	1.16	1	1.94
	10	11.5	10	12.9
$\Delta T_0/\tau_{nom}$	1	6.98	1	7.00
	10	36.5	10	39.8

[a] $\Delta x_i/x_{i\,nom}\cdot 100\%$.
[b] $[(\Delta\alpha_{nonlin} - \Delta\alpha_{lin})/\Delta\alpha_{nonlin}]\cdot 100\%$.

greater accuracy than is possible on the basis of individual horizon sensor measurements. Conceptually, this is done as follows: at any time t, $\alpha_{\text{nom}}(t)$ is calculated using the nominal set of orbital parameters, and a correction term is then added on the basis of the present best estimate of $\mathbf{\Delta x}$:

$$\hat{\alpha}(t) = \alpha_{\text{nom}}(t) + \mathbf{h}^T(t)\mathbf{\Delta\hat{x}} \tag{21}$$

If $\mathbf{\Delta\hat{x}}$ is small so that no nonlinearity errors are introduced, the result is equivalent to calculating $\hat{\alpha}(t)$ on the basis of $\mathbf{\hat{x}} = \mathbf{x}_{\text{nom}} + \mathbf{\Delta\hat{x}}$. The true value of $\alpha(t)$ based upon the true $\mathbf{\Delta x}$ is given by

$$\alpha(t) = \alpha_{\text{nom}}(t) + \mathbf{h}^T(t)\mathbf{\Delta x} \tag{22}$$

The error in the predicted direction of local vertical

$$\Delta\hat{\alpha}(t) = \alpha(t) - \hat{\alpha}(t) = \mathbf{h}^T(t)[\mathbf{\Delta x} - \mathbf{\Delta\hat{x}}] \tag{23}$$

is then a Gaussian random variable with the following mean and variance [since $\mathcal{E}(\mathbf{\Delta\hat{x}}) = \mathbf{\Delta x}$]:

$$\mathcal{E}[\Delta\hat{\alpha}(t)] = 0 \tag{24}$$

$$\sigma^2_{\text{est}}(t) \equiv \mathcal{E}[\Delta\hat{\alpha}(t)^2] = \mathbf{h}^T(t)\text{cov}(\mathbf{\Delta x} - \mathbf{\Delta\hat{x}})\mathbf{h}(t) \tag{25}$$

$$= \mathbf{h}^T(t)\mathbf{P}\mathbf{h}(t)\cdot\sigma^2$$

$\Delta\hat{\alpha}(t)$ has a mean of zero and a variance that is a function of the point on the orbit at which the determination is made, through $\mathbf{h}(t)$, as well as the confidence placed in the estimate $\mathbf{\Delta\hat{x}}$ as expressed through the $\mathbf{P}$ matrix. $\mathbf{P}$ depends upon the number and frequency of measurements made as well as the a priori knowledge expressed through $\mathbf{P}(0)$, but is independent of the actual measurement noise. $\mathbf{P}$ and $\mathbf{h}$ are functions of the nominal orbit, but only vary slightly in the linear region about a nominal orbit.

On the basis of $\mathbf{P}$ and $\mathbf{h}$, one can define the ratio $r(t)$ as follows:

$$r(t) \equiv \frac{\sigma_{\text{est}}(t)}{\sigma_{\text{noise}}} = [\mathbf{h}^T(t)\mathbf{P}\mathbf{h}(t)]^{1/2} \tag{26}$$

This ratio shows the improvement in the accuracy of the prediction of local vertical, which is achieved by using the best estimate of the orbital parameters as compared to the prediction based upon individual horizon sensor measurements. For a given nominal orbit and $\mathbf{P}(0)$, it is possible to compute the number of measurements (M) required to bring $r(t)$ below a given value, say 10^{-1}, for all points on the orbit. One may then say that, in a stochastic sense, the determination of local vertical from $\mathbf{\Delta\hat{x}}(M)$ is better by at least a factor of 10 than the determination from horizon sensor measurements. Similarly, the trace of $\mathbf{P}$ provides a measure of the accuracy of the predicted $\mathbf{\Delta x}$ since it determines the mean-square value of the errors in the parameters:

$$\text{tr}\mathbf{P}\cdot\sigma^2 = \mathcal{E}\left[\sum_{i=1}^{N}(\Delta x_i - \Delta\hat{x}_i)^2\right] \tag{27}$$

5. Computer Program and Results

A program has been written for the IBM 7090 Computer to simulate the motion of the vehicle as well as the estimation scheme. A simplified flow diagram is shown in Fig. 2. The initial version of the program uses the equations for a two-body orbit in determining α; however, these may easily be modified to include slow secular perturbations in the parameters, as will be seen below.§ The partials that determine $h(t)$ are always based upon the two-body orbit equations since their variations with changes in the nominal parameters is a second-order effect. One important feature

§ The program may also be modified to include disturbance torques if a numerical integration routine is introduced. See previous footnote.

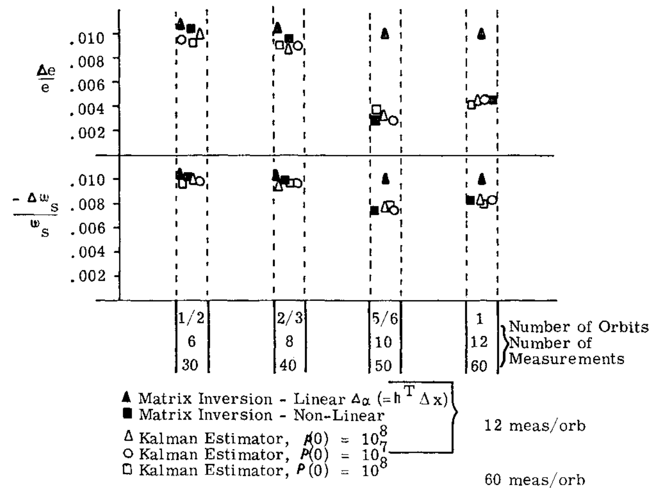

Fig. 3 Comparison of true and estimated parameter errors at four points on the orbit.

of the program is not shown in Fig. 2. This is the updating feature. After a given number of measurements, say K, $\mathbf{x}_{\text{nom}}$ is updated to include the present best estimate of $\mathbf{\Delta x}$ and the time is reset to zero. Suitable changes are introduced in the parameters T_0 and α_b in the simulation and estimation scheme so that the point of updating appears as a new injection point for all future measurements. The arbitrary reference direction is taken as the direction of the best estimate of local vertical at the time of updating. For future measurements, $\mathbf{\Delta\hat{x}}(0) = 0$. If the linearity assumptions hold for $\mathbf{\Delta x}$ before and after updating, then $\mathbf{P}(K)$ before updating may be chosen as $\mathbf{P}(0)$ for future measurements. This is equivalent to the assumption that $\mathbf{h}(t)$ is not affected by the shift in the nominal set of parameters and that the confidence in the present best estimate of $\mathbf{x}$ and the direction of local vertical has not changed. In many cases, $\mathbf{P}(0)$ is reinitialized to some large value and the estimation of the parameters is begun anew.

The program was tested initially without any noise in the measurements, with $\mathbf{\Delta\hat{x}}(0) = 0$ and large initial $\mathbf{P}$ matrix.¶ This is equivalent to using the recursion formulas to obtain a least-squares fit to the measurement data. Theoretically, a set of N independent measurements should yield a perfect estimate of the errors in N parameters. This did not occur in practice because of the nonlinearities in the measured value of $\alpha(t)$. This is illustrated clearly in Fig. 3 for normalized

¶ Care must be taken that the initial $\mathbf{P}$ matrix is not made too large; otherwise numerical significance will be lost as the trace of $\mathbf{P}$ decreases in magnitude and the $\mathbf{P}$ matrix will no longer remain positive definite. This difficulty was encountered in the early phases of the investigation, especially when many measurements were taken over one orbit (100 or more). Eventually, it was found that an initial $\mathbf{P}$ matrix with diagonal terms anywhere between 10^3 and 10^5 led to the same results after the first few measurements for the particular trajectories considered, and the $\mathbf{P}$ matrix remained positive definite. Several variations of the recursion formula [Eq. (15)] are available[11,12] which guarantee that the $\mathbf{P}$ matrix remain nonnegative definite throughout the computation. The technique of Ref. 11 applies to single scalar measurements, as in Eq. (15), whereas the formula of Ref. 12 is valid for vector measurements as well. In our notation, the formula of Ref. 12 reduces to

$$\mathbf{P}(K) = \{\mathbf{I} - \mathbf{s}(K)\mathbf{h}^T(K)\}\mathbf{P}(K-1)\{\mathbf{I} - \mathbf{s}(K)\mathbf{h}^T(K)\}^T + \mathbf{s}(K)\mathbf{s}^T(K)$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{s}(K) = \mathbf{P}(K-1)\mathbf{h}(K)\{\mathbf{h}^T(K)\mathbf{P}(K-1)\mathbf{h}(K) + 1\}^{-1}$$

The preceding formula can be shown to be equivalent to Eq. (15).

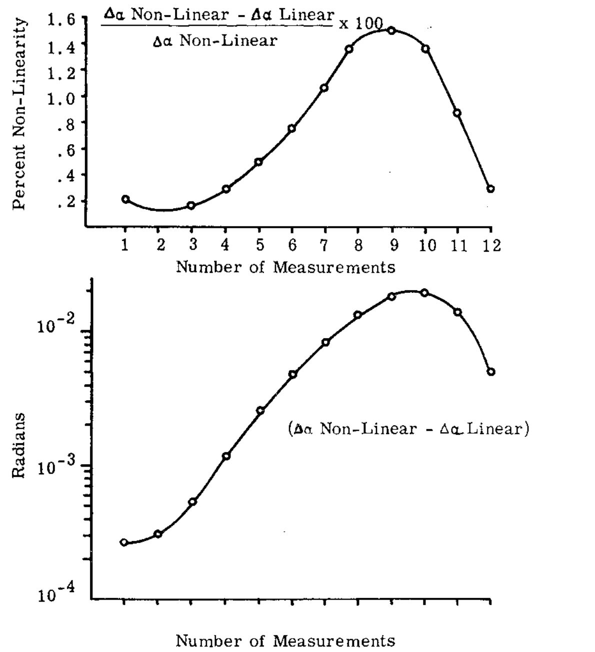

Fig. 4 **Nonlinear contribution to $\Delta\alpha$ along one orbit.**

errors of 1%. It is seen that the predicted value of Δx_i depends much more strongly on the portion of the orbit at which measurements have been taken than on the number of measurements or the initial **P** matrix chosen. When measurements are included at points on the orbit where nonlinearities are largest, the prediction of $\mathbf{\Delta x}$ deteriorates, as seen from Fig. 4. The magnitude of the nonlinearity will depend upon the coupling between the unknown components of $\mathbf{\Delta x}$ and cannot be determined beforehand in the estimation scheme. As long as updating reduces the size of the Δx_i, the linear region will be approached through updating and corrections will become more and more accurate. This is illustrated in Fig. 5 for two rates of updating, starting with the same set of nominal and actual orbital parameters (e = 0.101, e_{nom} = 0.1; a = 1.484, a_{nom} = 1.16; this corresponds to a perigee of 500 naut mile). When no noise is present in the measurements, rapid updating is advantageous. The **P** matrix is reset to its initial value at the time of updating in order to make full use of the next set of measurements.

Runs were also made for low-eccentricity cases with no noise in the measurements. A run with $e = 0$, and e_{nom} = 0.01 converged in e and $\Delta\alpha$ (i.e., both were less than 10^{-6} after five updatings), although the predicted values of T_0, ω_s, and a did not approach the true values. However, the errors in ω_s and a were such that $\hat{\omega}_s - 2\pi/\hat{\tau} \approx \omega_s - 2\pi/\tau$, which is related to the fact that ω_s and a are no longer independent observables on a circular orbit.

When noise is introduced into the measurements, the behavior of the system will depend upon whether the difference between the predicted angle α_{nom} and the measured angle α_m is primarily due to errors in the parameter values, or whether the difference due to parameter errors is of the same order of magnitude as the noise. This is illustrated in Fig. 6. The no-noise case with updating once per orbit of Fig. 5 is compared to two cases with the same initial conditions and rate of updating, but different values of σ, the standard deviation of the noise. The first few updatings improve the prediction of α, as in the no-noise case. The improvement does not extend below the order of magnitude of σ, however, for continued updating after 60 measurements and one orbit. With 120 measurements per orbit, the improvement in α eventually exceeds that of the case with 60 measurements, although the initial updating did not happen to be as good for the particular noise samples chosen. The explanation for these results may be found by studying the behavior of the **P** matrix, its trace, and the quantity $\mathbf{h}^T\mathbf{P}\mathbf{h}$. Figure 7 is a plot of the tr**P** vs number of orbits for the nominal case with 60 measurements per orbit, and 120 measurements per orbit, respectively. For a given number of measurements, the trace decreases more rapidly when the measurements are further apart in time. For a given time, however, the trace is smaller when more measurements have been taken. At the time of updating, the trace of **P** multiplied by the variance of the noise determines the mean square values of the errors in the parameters x_i [from Eq. (27)]. This provides an indirect measure of the mean square error to be expected in the prediction of α based on the new set of parameters. A more direct indication is provided by the

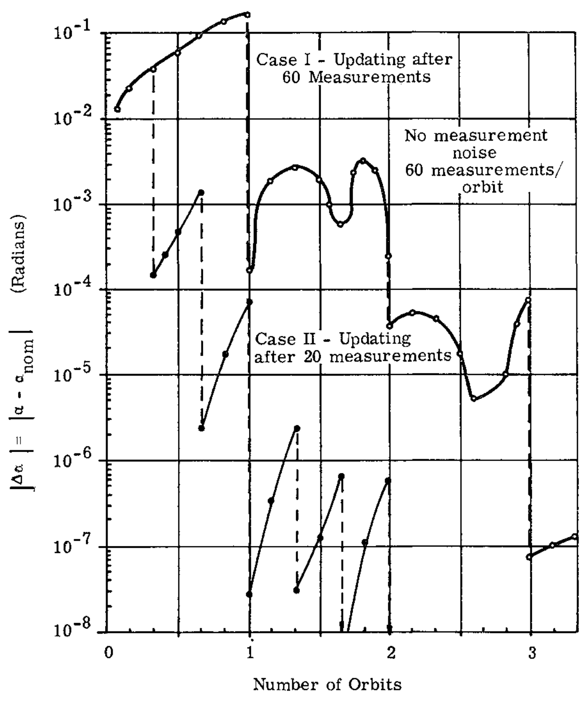

Fig. 5 **Reduction of $\Delta\alpha$ for two rates of updating.**

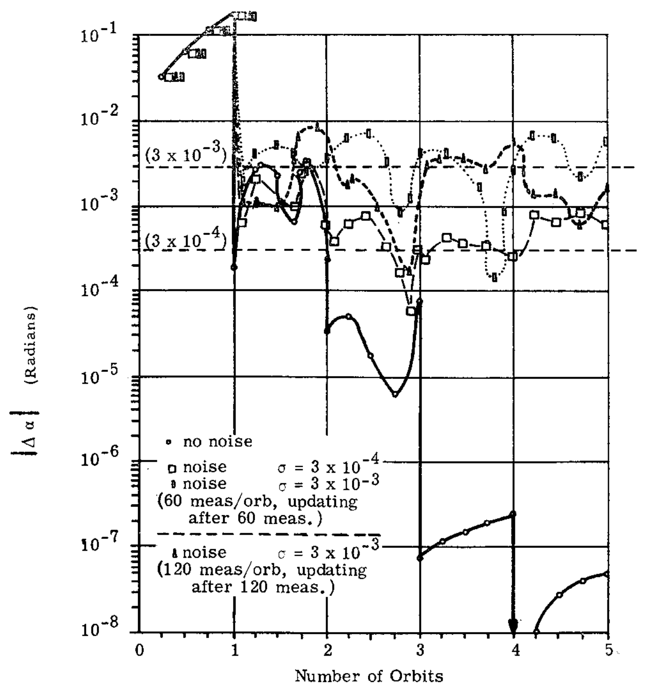

Fig. 6 **Reduction of $\Delta\alpha$ for various noise samples.**

quantity $\mathbf{h}^T\mathbf{Ph}$. Figure 8 compares the tr$\mathbf{P}$ to the quantity $\mathbf{h}^T\mathbf{Ph}$ for the case of 120 measurements per orbit. After 120 measurements, the quantity $r = (\mathbf{h}^T\mathbf{Ph})^{1/2}$ has a value 0.5. If the nominal orbital parameters are updated after each set of 120 measurements and the $\mathbf{P}$ matrix is reinitialized, as in Fig. 6, then the error in the predicted angle cannot be expected to fall below 10^{-3} when the measurement noise has a σ of 3×10^{-3}.

In order to improve the ratio r at the time of updating, three methods may be employed.

1) The number of measurements per orbit may be increased. Besides the physical limitations on the system in taking and processing more measurements, the assumption of white noise becomes more difficult to justify as the number of measurements is increased.

2) The time between updatings may be increased. This method is limited by the tendency of $\Delta\alpha = \alpha - \alpha_{\text{nom}}$ to increase with time from updating (because of the increasing importance of errors in a and ω_s with time) and by the increasing importance of perturbations in the orbital parameters as the time from updating becomes large.

3) The orbital parameters may be updated at a faster rate than the $\mathbf{P}$ matrix; e.g., the orbital parameters may be updated once every orbit but the $\mathbf{P}$ matrix reinitialized every other orbit. This allows for twice as many measurements as otherwise to be included in determining the $\hat{x}_i$ on which the angle α is to be based.

From the test cases run, it appears that a compromise between methods 1 and 2 is required for optimum performance. For the first few updatings, when errors in the x_i may be large, it is desirable to take as many measurements as possible over a short interval of time and then update and reinitialize the procedure in order to reduce the nonlinear effects and prevent $\Delta\alpha$ from growing with time. Once $\Delta\alpha$ is within the noise region, a greater reduction in r is achieved by spreading the same number of measurements over a longer time interval. As the number of measurements and the length of the interval is increased, a value is reached beyond which improvement is hardly noticeable, as the examples below will illustrate. When $\Delta\alpha$ is within the noise level, method 3 may have some advantages over method 2 in tracking slow perturbations in the elements, assuming that orbital elements are updated at a faster rate in method 3.

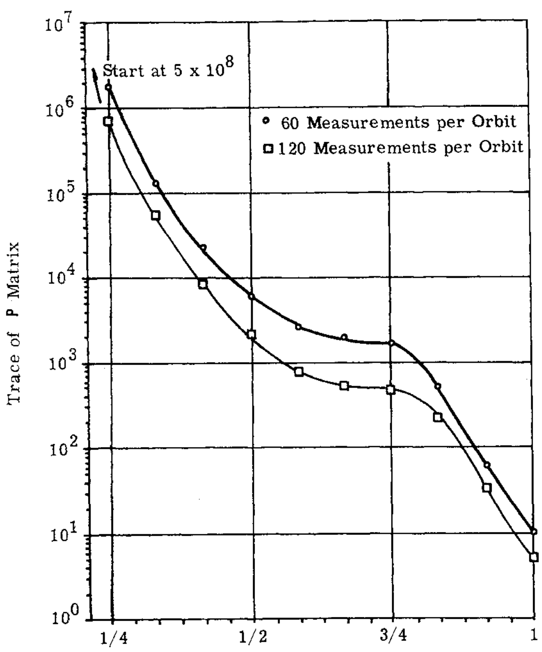

Fig. 7 Trace of P matrix vs number of measurements.

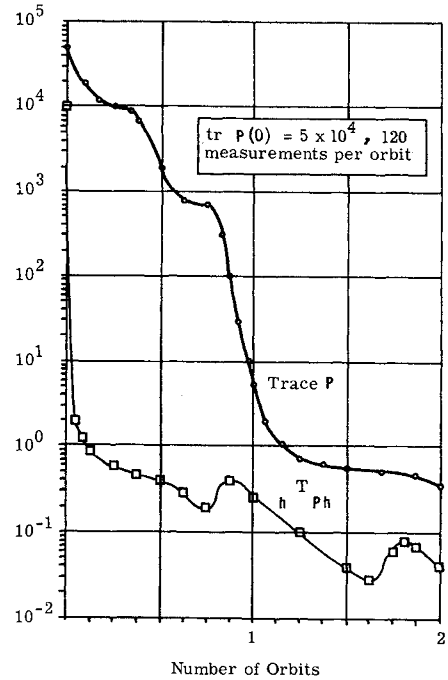

Fig. 8 Comparison of trP and $\mathbf{h}^T$ P h.

For constant elements, however, method 2 has been consistently better.

In the examples that follow, measurement noise in the horizon sensor with a standard deviation of $\sigma = 10^{-2}$ ($\approx 0.6°$ of arc) was assumed. The results apply for any smaller value since only ratios are involved. It was found that, in order to reduce r to 0.10 for the orbital parameters assumed in the previous examples, at least 700 measurements are required when 200 measurements per orbit are taken (3.5 rev), over 750 measurements are required with 500 measurements per orbit (1.5 rev), and 1300 measurements are required for 1000 measurements per orbit (1.3 rev). Figures 9 and 10 show the quantities $(\mathbf{h}^T\mathbf{Ph})^{1/2}\cdot\sigma$ and $(\text{tr}\mathbf{P})^{1/2}\cdot\sigma$ after one orbital

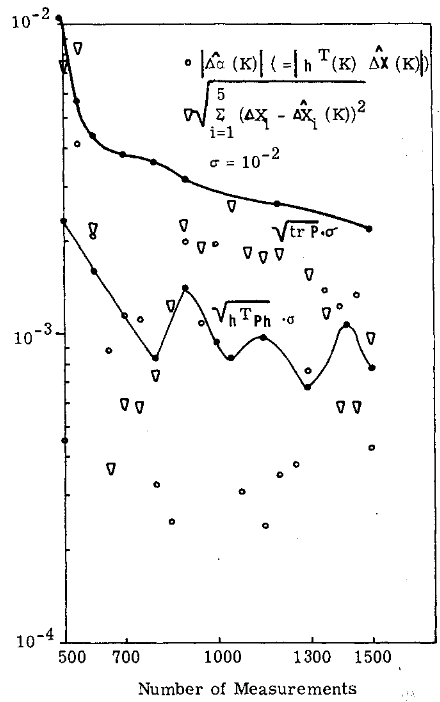

Fig. 9 Comparison of predicted rms values of errors and actual errors for a run with 500 measurements/orbit.

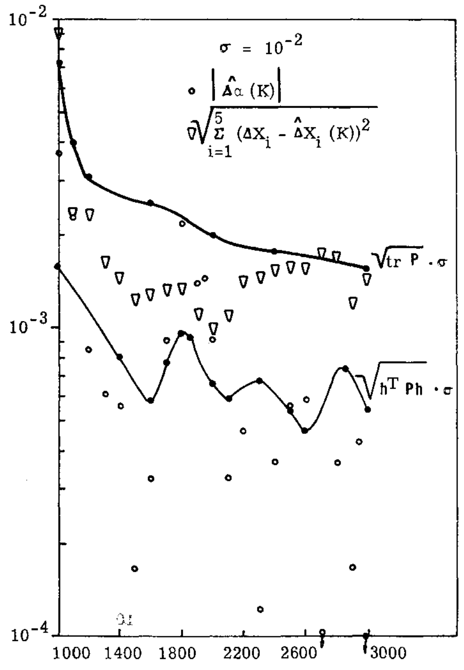

Fig. 10 Comparison of predicted rms values of errors and actual errors for a run with 1000 measurements/orbit.

revolution for the cases of 500 measurements per orbit and 1000 measurements per orbit, respectively. Scattered about these curves are shown some of the actual values of

$$|\Delta\hat{\alpha}(K)| \text{ and } \left[\sum_{i=1}^{5} [\Delta x_i - \Delta\hat{x}_i(K)]^2\right]^{1/2}$$

for a particular run, with the initial errors given by

$$\left[\sum_{i=1}^{5} (\Delta x_i)^2\right]^{1/2} = 0.591 \times 10^{-2} \qquad \Delta\hat{x}_i(0) = 0$$

and an initial **P** matrix with a trace of 5×10^4. (The initial **P** matrix may be changed by a factor of 100 or more without changing the nature of the results past the first half of an orbit.) For the runs illustrated, the errors in the orbital parameters based upon the predicted $\mathbf{\Delta\hat{x}}(K)$ are consistently better than the rms values determined from tr**P**. The values of $\Delta\hat{\alpha}$ also fall mostly below the curve of $(\mathbf{h}^T\mathbf{Ph})^{1/2}\cdot\sigma$. From the curves, it is seen that the rate of improvement in tr**P** becomes slower as the number of measurements is increased. The reduction of $r = (\mathbf{h}^T\mathbf{Ph})^{1/2}$ much below 0.10 becomes exceedingly expensive in terms of the number of measurements required and/or the number of orbits required between updatings.

With 500 measurements per orbit, updating near 800 measurements appears reasonable from Fig. 9. Figure 11 shows the behavior of $|\Delta\alpha| = |\alpha - \alpha_{\text{nom}}|$ for a particular run in which the parameters were updated twice after 700 measurements and the third time after 800 measurements. The **P** matrix was reinitialized at the time of updating. Between updatings, the nominal values of the parameters at the most recent updating are used in determining α_{nom}. The predicted rms value of $|\Delta\alpha|$ between updatings are also shown. These are based upon the value of the **P** matrix just prior to the most recent updating and reflect the confidence placed in the most recent set of nominal parameters. Changes in the predicted rms value about the orbit are due to the changes in $\mathbf{h}(t)$ about the orbit. The initial errors, before any updating, are the same as in Figs. 5 and 6. After two updatings, the actual values of $|\Delta\alpha|$ fall mostly below 10^{-3}, a factor of 10 improvement over σ, and are smaller than the predicted rms values at most points on the orbit.

6. Effects of Perturbations

The orbital parameters that determine the translational motion of an earth satellite are not true constants, as they would be for simple Keplerian motion, but are subject to variation due to the perturbing effects of the earth's atmosphere, the earth's oblateness, and the gravitational attraction of the moon and the planets. For orbits under consideration in this paper, atmospheric effects are negligible, and the primary source of perturbations is the oblateness of the earth. The type of perturbations that occur may be classified as periodic or secular, depending upon whether they are cyclical, with a period of one or several orbital revolutions, or monotonic over many orbits, respectively. The nature of the periodic perturbations has been investigated using the results of Kozai.[13, 14] It has been shown that the perturbations in each of the parameters a and e have maximum amplitude on the order of A_2, where A_2 is the constant in the first correction term for the gravitational potential of the earth due to oblateness. A recently determined value for A_2 is 3.2×10^{-3}. For small eccentricities, the maximum perturbation in T_0 may cause variations in the mean anomaly and the true anomaly which are much larger than A_2; however, these are balanced by variations in ω, the argument of perigee. With perturbed orbital parameters, the direction of local vertical must be determined from the argument of latitude $L = \varphi + \omega$, where φ is the true anomaly. Reference 14 shows that perturbations in L never exceed A_2 in magnitude. For orbits most sensitive to perturbations, the maximum amplitude of the periodic perturbations is still an order of magnitude less than $\sigma = 0.01$, the standard deviation of the measurement noise. Periodic perturbations may therefore be neglected.

As far as secular perturbations are concerned, Kozai[14] and others[15] have shown that there are no secular perturbations of first- or second-order in the parameters a, e, or i (the inclination angle of the orbit with respect to the equatorial plane). Secular perturbations in T_0 [or equivalently, in the

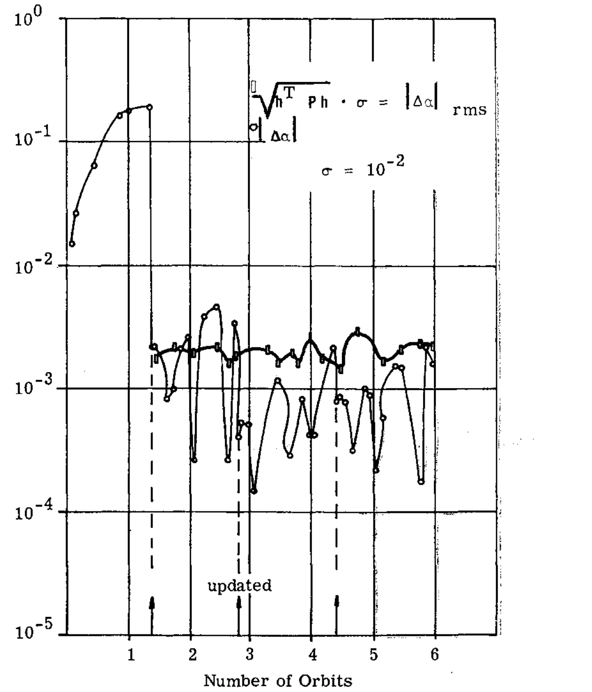

Fig. 11 Reduction of $\Delta\alpha$ for a case with 500 measurements/orbit.

mean anomaly, through Eq. (4)] must be considered, however. To account for this perturbation, the value of T_0 in the simulation program was increased at each measurement time by an amount ΔT_0 given by the following equation, where Δt is the time interval between measurements[15]:

$$\Delta T_0 = -\{A_2/[a^2(1 - e^2)^{3/2}]\}(\tfrac{3}{2}\sin^2 i - 1)\Delta t \qquad (28)$$

For an inclination angle $i = 0$, and the orbital parameters of the previous examples ($e = 0.1$, $a = 1.1484$), this leads to a change in T_0 of about 0.26 min/orbital rev. When a corresponding change is added to the calculation of α_{nom} in the estimation scheme, the error in the predicted angle remains essentially the same as with the unperturbed parameters. This had been demonstrated for runs with and without noise.

In adding the perturbation to the estimation scheme, a_{nom} and e_{nom} are used in Eq. (28) and the value of ΔT_0 calculated from this equation is added to $T_{0\,\text{nom}}$ in Eq. (4) at each measurement time. This determines a new value of $M(t)$ which is then used in the determination of $\alpha_{\text{nom}}(t)$ through Eqs. (2) and (3). For the determination of the partial derivatives in $\mathbf{h}(t)$, however, the initial unchanged value of $T_{0\,\text{nom}}$ is used, until the time is reached at which all the parameters are updated.

After many orbital revolutions, long term periodic perturbations due to oblateness and other disturbing forces besides oblateness will cause changes in the parameters. If the procedure is reinitialized after every two or three orbital revolutions, the scheme should follow these changes over long periods of time, and the parameters may be considered as constant between updatings. A full-scale simulation including many of these disturbing forces must be performed to test the long term behavior of the estimation scheme. This simulation would also include the effects of rotations about axes parallel to the orbital plane. If the true axis of rotation makes a small angle with the normal to the orbital plane, then out-of-plane motions are independent of motions in the orbital plane to first order.

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